

Operator reflection positivity inequalities and their applications to interacting quantum rotors

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Abstract

In the Reflection Positivity theory and its application to statistical mechanical systems, certain matrix inequalities play a central role. The Dyson-Lieb-Simon [1] and Kennedy-Lieb-Shastry [2] inequalities constitute prominent examples. In this paper we extend the KLS inequality to the case where matrices are replaced by certain operators. As an application, we prove the occurrence of the long range order in the ground state of two-dimensional quantum rotors.

Keywords: Statistical mechanics; phase transitions; operator inequalities; reflection positivity

1 Introduction

The *Reflection Positivity* notion has appeared in Quantum Field Theory in seventies of the last century [3]. Few years later, it has been applied to investigation of phase transitions in both classical [4] and quantum [1] lattice spin systems. The Reflection Positivity turned out to be a very useful tool, giving the first rigorous proofs of existence of phase transitions in systems with continuous symmetry group.

The cornerstone of Reflection Positivity for quantum spin systems is the matrix inequality due to Dyson, Lieb and Simon (Lemma 4.1 in [1]). Using this Lemma, authors proved the existence of orderings in the XY as well as Heisenberg models in $d \geq 3$ and for sufficiently small temperature. Later on, this method has been extended to certain class of *infinite dimensional* operators. This way, the existence of Long-Range Order has been proved for $d \geq 3$ in the system of quantum interacting rotors [5].

Another direction of development of Reflection Positivity techniques was an examination of *ground states* of quantum spin systems and orderings therein. It turned out that one can take certain zero-temperature limit in the framework of the DLS method. This way, the appearance of Long Range Order has been proved in XY and Heisenberg models in $d = 2$ [6], [7], [8]. Later on, it turned out that such a proof can be done directly in the ground state, with the use of another matrix inequality, due to Kennedy, Lieb and Shastry (KLS) [2]. This inequality was further generalized by Schupp [9].

It would be tempting to extend this inequality to infinite-dimensional version, i.e. for certain class of operators. However, to our best knowledge, the operator version of the KLS and Schupp(KLSS) inequalities, suitable for applications to ground states of quantum interacting rotors has *not* been developed.

This opportunity inspired us to attempts to prove an operator analog of the KLSS inequalities. It turned out to be possible, and this is one of two main results of our paper: *extension of the KLSS matrix inequalities to certain class of infinite-dimensional operators*. The second group of results which seems to be new are some applications.

The outline of the paper is as follows. In the Sec. 2 we formulate the operator version of the KLSS inequalities. The application of this inequality is described in the Sec. 3; it is the proof of the ordering in ground state of $d \geq 2$ rotors (alternative proof to that presented in [10]). The Sec. 4 contains summary, conclusions and description of some open problems.

2 KLS inequality and its extension for operators

2.1 Kennedy, Lieb, Shastri and Schupp matrix inequalities.

For convenience of the reader, and to show the idea of a proof without operator-theoretic details, we present firstly the matrix version of KLSS inequality.

Theorem 2.1 [2] *Let c, A, B be $n \times n$ complex matrices, $|c| := \sqrt{c^* c}$ and $|c^*| := \sqrt{c^* c^*}$ the moduli of c and c^* respectively. Then*

$$|\text{Tr } c^* B c A^*| \leq \frac{1}{2} [\text{Tr } (|c| A |c| A^*) + \text{Tr } (|c^*| B |c^*| B^*)]. \quad (1)$$

Sketch of the proof: At first let us note that by the polar decomposition theorem c is of the form $c = u |c|$, where u is a partial isometry. Since $u^* u |c| = |c|$ and $u |c| u^*$ is a positive matrix, the polar decomposition of c^* is of the form:

$$c^* = |c| u^* = u^* u |c| u^* = u^* |c^*|. \quad (2)$$

Taking adjoint we get $c = u |c| = |c^*| u$. Therefore

$$u \sqrt{|c|} = \sqrt{|c^*|} u \quad (3)$$

according to functional calculus of positive hermitian matrices and

$$c = \sqrt{|c^*|} u \sqrt{|c|}. \quad (4)$$

Now, let P and Q be matrices introduced by formulae

$$P = u^* \sqrt{|c^*|} B^* \sqrt{|c^*|} u, \quad Q = \sqrt{|c|} A^* \sqrt{|c|}. \quad (5)$$

Let us remind that the trace functional defines the scalar product on the space of square matrices: $(A|B) := \text{Tr } A^* B$. Therefore

$$|\text{Tr } P^* Q| \leq \sqrt{(\text{Tr } P^* P) (\text{Tr } Q^* Q)} \leq \frac{1}{2} [\text{Tr } P^* P + \text{Tr } Q^* Q] \quad (6)$$

due to the Schwarz inequality followed by inequality between geometric mean and arithmetic one. Now using formula (3) and (4) one can easily verify that

$$P^* Q = u^* \sqrt{|c^*|} B \sqrt{|c^*|} u \sqrt{|c|} A^* \sqrt{|c|} = \sqrt{|c|} u^* B c A^* \sqrt{|c|}$$

and due to (2)

$$\text{Tr } P^* Q = \text{Tr } \left(\sqrt{|c|} u^* B c A^* \sqrt{|c|} \right) = \text{Tr } (|c| u^* B c A^*) = \text{Tr } (c^* B c A^*).$$

The module of it coincides with the left hand side of the inequality (1).

In the similar manner we compute the right hand side of (6):

$$P^* P = u^* \sqrt{|c^*|} B \sqrt{|c^*|} u u^* \sqrt{|c^*|} B^* \sqrt{|c^*|} u = u^* \sqrt{|c^*|} B |c^*| B^* \sqrt{|c^*|} u.$$

Therefore

$$\text{Tr } P^* P = \text{Tr } \left(u^* \sqrt{|c^*|} B |c^*| B^* \sqrt{|c^*|} u \right) = \text{Tr } (|c^*| B |c^*| B^*). \quad (7)$$

By the similar reasoning we get

$$\text{Tr } Q^* Q = \text{Tr } \left(\sqrt{|c|} A |c| A^* \sqrt{|c|} \right) = \text{Tr } (|c| A |c| A^*). \quad (8)$$

Combining (7) and (8) we obtain the right hand side of inequality (1). ■

Theorem 2.2 [9] *The KLS inequality (1) holds also for rectangular matrices, i.e. c is $n \times m$ matrix and matrices A and B are $m \times m$ and $n \times n$ respectively.*

Proof: It is almost a repetition of the proof for KLS inequality and refers to the modified polar decomposition of the operator c acting between m -dimensional and n -dimensional spaces. In that situation $|c|$, $|c^*|$ are positive matrices of dimensions $m \times m$ and $n \times n$, respectively. ■

2.2 Operator version of the KLSS inequality

The main goal of this subsection is to prove a generalized version of the KLSS inequality:

Theorem 2.3 *Let \mathcal{L} and \mathcal{R} be separable Hilbert spaces, $A \in \mathcal{B}(\mathcal{L})$, $B \in \mathcal{B}(\mathcal{R})$ bounded operators acting on \mathcal{L} and \mathcal{R} respectively. Let $c : \mathcal{L} \rightarrow \mathcal{R}$ be a Hilbert-Schmidt operator and $|c| := \sqrt{c^*c}$, and $|c^*| := \sqrt{c c^*}$ be the corresponding moduli. Then*

1. $|c|$ and $|c^*|$ are hermitean Hilbert-Schmidt operators acting on \mathcal{L} and \mathcal{R} respectively;
2. $c^* B c A^*$, $|c| A |c| A^*$ are trace-class operators on \mathcal{L} and $|c^*| B |c^*| B^*$ is a trace-class operator on \mathcal{R} ;
3. the following inequality holds

$$|\mathrm{Tr} \ c^* B c A^*| \leq \frac{1}{2} [\mathrm{Tr} (|c| A |c| A^*) + \mathrm{Tr} (|c^*| B |c^*| B^*)] \quad (9)$$

Remark. In a finite-dimensional case, i.e. $\mathcal{L} = \mathbb{C}^N = \mathcal{R}$ the above inequality reduces to the matricial KLS inequality [2] and in more general finite dimensional situation $\dim \mathcal{L} \neq \dim \mathcal{R}$ we obtain the result of Schupp [9].

Proof: To prove our result we shall use some properties of *Schatten ideals* [17], [18], [19] and now we shall recall necessary results of the theory.

Let \mathcal{H} be a separable Hilbert space, $\mathcal{CB}(\mathcal{H})$ - the set of compact operators on \mathcal{H} . For a real number $p \geq 1$ the p -Schatten ideal is the set

$$L^p(\mathcal{H}) := \{a \in \mathcal{CB}(\mathcal{H}) : \mathrm{Tr} (|a|^p) < \infty\}.$$

Remark. Sometimes $L^p(\mathcal{H})$ is denoted as \mathcal{J}_p [14], however, the actual notation corresponding to noncommutative L^p -spaces seems more natural to us.

For $a \in L^p(\mathcal{H})$ let us define:

$$\|a\|_p := (\mathrm{Tr} |a|^p)^{\frac{1}{p}}.$$

Then it is known that

- i) $(L^p(\mathcal{H}), \|\cdot\|_p)$ is a Banach space, $L^p(\mathcal{H})$ is a two-sided ideal in $\mathcal{B}(\mathcal{H})$, i.e. for any $a \in L^p(\mathcal{H})$ and $A, B \in \mathcal{B}(\mathcal{H})$ the operator $AaB \in L^p(\mathcal{H})$ and moreover

$$\|AaB\|_p \leq \|A\| \|B\| \|a\|_p.$$

- ii) if $p, q, r \geq 1$ are such numbers that: $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ and $a \in L^p(\mathcal{H})$, $b \in L^q(\mathcal{H})$, then $ab \in L^r(\mathcal{H})$.

- iii) in particular, if p and q satisfy $\frac{1}{p} + \frac{1}{q} = 1$ then for $a \in L^p(\mathcal{H})$, $b \in L^q(\mathcal{H})$ products ab , $ba \in L^1(\mathcal{H})$,

$\mathrm{Tr} \ ab = \mathrm{Tr} \ ba$ and

$$|\mathrm{Tr} \ ab| \leq \|a\|_p \|b\|_q.$$

Remark. The space $L^1(\mathcal{H})$ is the space of trace-class operators on \mathcal{H} and $L^2(\mathcal{H})$ is the space of Hilbert-Schmidt class. Clearly $L^2(\mathcal{H})$ equipped with the sesquilinear form

$$L^2(\mathcal{H}) \times L^2(\mathcal{H}) \ni (a, b) \longmapsto (a|b) := \mathrm{Tr} \ a^* b \in \mathbb{C}$$

is a Hilbert space.

In what follows we shall also need Hilbert-Schmidt operators in more general settings, namely the operators from one Hilbert space to another.

Definition. An operator $c : \mathcal{L} \rightarrow \mathcal{R}$ is a Hilbert-Schmidt one, if for some orthonormal basis $\{\alpha\}$ in \mathcal{L} , the sum

$$\|c\|_2^2 := \sum_{\alpha} (c\alpha|c\alpha) = \sum_{\alpha} \|c\alpha\|^2$$

is finite.

The set of such operators will be denoted by $L^2(\mathcal{L}, \mathcal{R})$. Clearly for $c \in L^2(\mathcal{L}, \mathcal{R})$ and any orthonormal basis $\{\beta\}$ in \mathcal{R} , we have

$$\begin{aligned} \sum_{\alpha} (c\alpha|c\alpha) &= \sum_{\alpha} (\alpha|c^*c\alpha) = \sum_{\alpha, \beta} |(\beta|c\alpha)|^2 \\ &= \sum_{\alpha, \beta} |(c^*\beta|\alpha)|^2 = \sum_{\beta} \|c^*\beta\|^2 = \sum_{\beta} (\beta|c c^*\beta). \end{aligned} \quad (10)$$

Therefore $c^* \in L^2(\mathcal{R}, \mathcal{L})$ and the finiteness condition does not depend on the particular choice of an orthonormal basis $\{\alpha\}$. In particular $c^*c \in L^1(\mathcal{L})$, $cc^* \in L^1(\mathcal{R})$ i.e. they are trace-class operators acting on \mathcal{L} and \mathcal{R} respectively. Moreover

$$\|c\|_2^2 = \|c^*c\|_1 = \text{Tr}_{\mathcal{L}}(c^*c) = \text{Tr}_{\mathcal{R}}(cc^*) = \|cc^*\|_1 = \|c^*\|_2^2. \quad (11)$$

Let us note that for $a, b \in L^2(\mathcal{L}, \mathcal{R})$ we have $a^*b \in L^1(\mathcal{L})$, $ba^* \in L^1(\mathcal{R})$. Moreover one can easily check that for any Hilbert spaces $\mathcal{L}', \mathcal{L}, \mathcal{R}, \mathcal{R}'$ a modified ideal property holds:

$$\left(\begin{array}{c} c \in L^2(\mathcal{L}, \mathcal{R}) \\ \text{and} \\ A \in \mathcal{B}(\mathcal{L}', \mathcal{L}), B \in \mathcal{B}(\mathcal{R}, \mathcal{R}') \end{array} \right) \Rightarrow (BcA \in L^2(\mathcal{L}', \mathcal{R}')). \quad (12)$$

As before the space $L^2(\mathcal{L}, \mathcal{R})$ forms a Hilbert space equipped with the scalar product

$$(a|b) := \text{Tr}_{\mathcal{L}}(a^*b) = \text{Tr}_{\mathcal{R}}(ba^*) = (b^*|a^*). \quad (13)$$

The last equality can be verified by the similar calculation as above. *In what follows to simplify notation the corresponding indices \mathcal{L} or \mathcal{R} will be omitted.* As the result, by the Schwarz inequality followed by mean arithmetic-geometric inequality, we obtain

Corollary 2.4 For arbitrary $a, b \in L^2(\mathcal{L}, \mathcal{R})$ we have

$$|\text{Tr } a^*b| \leq \frac{1}{2} [\text{Tr } a^*a + \text{Tr } b^*b] \quad (14)$$

Now we are ready to prove our result. Assume that $c \in L^2(\mathcal{L}, \mathcal{R})$. Therefore c^*c and cc^* are trace-class i.e. $|c| \in L^2(\mathcal{L})$ and $|c^*| \in L^2(\mathcal{R})$ and this proves the first part of the theorem. The second part easily follows from (12).

To prove the inequality let us note that by the polar decomposition theorem c is of the form $c = u|c|$ for the unique partial isometry $u \in \mathcal{B}(\mathcal{L}, \mathcal{R})$, $u : |c|(\mathcal{L}) \rightarrow c(\mathcal{L})$ such that u^*u and uu^* are projections on the initial and final domain respectively. Now, by uniqueness of the polar decomposition and functional calculus of bounded, self-adjoint operators we obtain (in the same way as for matrices):

$$u\sqrt{|c|} = \sqrt{|c^*|}u. \quad (15)$$

Therefore

$$c = \sqrt{|c^*|}u\sqrt{|c|}. \quad (16)$$

Clearly $\sqrt{|c|} \in \mathcal{L}^4(\mathcal{L})$ and $\sqrt{|c^*|} \in \mathcal{L}^4(\mathcal{R})$ and this observation enables us to follow the proof given for matrices in [2], [9]. As in (5) we define operators $P, Q \in \mathcal{B}(\mathcal{L})$:

$$P = u^*\sqrt{|c^*|}B^*\sqrt{|c^*|}u, \quad Q = \sqrt{|c|}A^*\sqrt{|c|}. \quad (17)$$

Remembering that L^p -spaces are ideals and using property *ii*) of Schatten ideals (for $p = q = 4$) we see that $Q \in L^2(\mathcal{L})$ and $\sqrt{|c^*|}B^*\sqrt{|c^*|} \in L^2(\mathcal{R})$; by (12) we have $P \in L^2(\mathcal{L})$. Now (14) reads

$$|\text{Tr } P^*Q| \leq \frac{1}{2} [\text{Tr } P^*P + \text{Tr } Q^*Q] \quad (18)$$

To compute the left hand side of the above expression let us notice that $u^* \sqrt{|c^*|} = \sqrt{|c|} u^*$ by formula (15). Using this fact and (16) we get

$$P^* Q = u^* \sqrt{|c^*|} B \sqrt{|c^*|} u \sqrt{|c|} A^* \sqrt{|c|} = \sqrt{|c|} u^* B c A^* \sqrt{|c|}.$$

The operator $u^* B c A^*$ belongs to $L^2(\mathcal{L})$ due to the modified ideal property (12). Since $\sqrt{|c|} \in L^4(\mathcal{L})$ the operator $\sqrt{|c|} u^* B c A^* \in L^{\frac{4}{3}}(\mathcal{L})$ by property *ii*) of Schatten ideals. Now using property *iii*) in the case $p = \frac{4}{3}$ and $q = 4$ we have

$$\text{Tr } P^* Q = \text{Tr } \left(\sqrt{|c|} u^* B c A^* \sqrt{|c|} \right) = \text{Tr } (|c| u^* B c A^*) = \text{Tr } (c^* B c A^*)$$

due to (2). The module of it coincides with the left hand side of the inequality (9). In the similar manner we compute the right hand side of (18).

$$P^* P = u^* \sqrt{|c^*|} B \sqrt{|c^*|} u u^* \sqrt{|c^*|} B^* \sqrt{|c^*|} u = u^* \sqrt{|c^*|} B |c^*| B^* \sqrt{|c^*|} u.$$

Therefore

$$\begin{aligned} \text{Tr } P^* P &= \text{Tr } \left(u^* \sqrt{|c^*|} B |c^*| B^* \sqrt{|c^*|} u \right) = \left\| \sqrt{|c^*|} B^* \sqrt{|c^*|} u \right\|_2^2 \\ &= \left\| u^* \sqrt{|c^*|} B \sqrt{|c^*|} \right\|_2^2 = \text{Tr } \left(\sqrt{|c^*|} B^* \sqrt{|c^*|} u u^* \sqrt{|c^*|} B \sqrt{|c^*|} \right) \\ &= \text{Tr } \left(\sqrt{|c^*|} B^* |c^*| B \sqrt{|c^*|} \right) \end{aligned}$$

by (11). Now $\sqrt{|c^*|} B^* \in L^4(\mathcal{R})$ and $|c^*| B \sqrt{|c^*|} \in L^{\frac{4}{3}}(\mathcal{L})$ and using property *iii*) again we obtain

$$\text{Tr } P^* P = \text{Tr } (|c^*| B |c^*| B^*). \quad (19)$$

By the similar reasoning we get

$$\text{Tr } Q^* Q = \text{Tr } \left(\sqrt{|c|} A |c| A^* \sqrt{|c|} \right) = \text{Tr } (|c| A |c| A^*). \quad (20)$$

Combining (19) and (20) we have the right hand side of inequality (9). The proof is done. \blacksquare

2.3 Main inequality and expectation values

In this subsection the main inequality (9) will be expressed in terms of expectation values of operators acting on $\mathcal{L} \otimes \mathcal{R}$, where \mathcal{L} and \mathcal{R} are separable, infinite dimensional Hilbert spaces. In that form it will be used in following sections.

Let $\Gamma := \{\psi_\gamma\}$ denote a fixed orthonormal basis in \mathcal{L} . It defines a linear map:

$$\tilde{\Gamma} : L^2(\mathcal{L}, \mathcal{R}) \ni c \mapsto \tilde{\Gamma}(c) = \sum \psi_\gamma \otimes c \psi_\gamma \in \mathcal{L} \otimes \mathcal{R}$$

Basic properties of this map are described by

Lemma 2.5 *The map $\tilde{\Gamma}$ is unitary, moreover for $B \in \mathcal{B}(\mathcal{R})$ and $a, c \in L^2(\mathcal{L}, \mathcal{R})$ we have:*

$$\begin{aligned} \tilde{\Gamma}(Bc) &= (I \otimes B) \tilde{\Gamma}(c) \\ \left(\tilde{\Gamma}(a) \mid (I \otimes B) \tilde{\Gamma}(c) \right) &= \left(\tilde{\Gamma}(a) \mid \tilde{\Gamma}(Bc) \right) = \text{Tr}_{\mathcal{L}}(a^* Bc) = \text{Tr}_{\mathcal{R}}(Bca^*) = \text{Tr}_{\mathcal{R}}(ca^* B) \end{aligned} \quad (21)$$

Proof: It is straightforward to verify that $\tilde{\Gamma}$ is an isometry i.e. for $c_1, c_2 \in L^2(\mathcal{L}, \mathcal{R})$ we have $\left(\tilde{\Gamma}(c_1) \mid \tilde{\Gamma}(c_2) \right) = \text{Tr}_{\mathcal{L}}(c_1^* c_2)$. The formula for $\tilde{\Gamma}(Bc)$ is clear. To show unitarity, notice that any $h \in \mathcal{L} \otimes \mathcal{R}$ is of the form $h = \sum \psi_\gamma \otimes r_\gamma$ for the unique family of vectors (r_γ) in \mathcal{R} . Let us define a linear map $\tilde{\Gamma}_1(h) : \mathcal{L} \rightarrow \mathcal{R}$ by $\tilde{\Gamma}_1(h) \psi_\gamma := r_\gamma$. Then $\tilde{\Gamma}_1(h) \in L^2(\mathcal{L}, \mathcal{R})$ and simple calculation shows that $\tilde{\Gamma} \tilde{\Gamma}_1(h) = h$ and $\tilde{\Gamma}_1 \tilde{\Gamma}(c) = c$, so $\tilde{\Gamma}$ is unitary and $\tilde{\Gamma}_1 = \tilde{\Gamma}^*$. The formula (21) is also clear; the last two equalities follow from properties of trace: $\text{Tr}_{\mathcal{L}}(a^* c) = \text{Tr}_{\mathcal{R}}(ca^*)$. \blacksquare

The basis Γ defines also an antiunitary involution $J_\Gamma : \mathcal{L} \rightarrow \mathcal{L}$:

$$J_\Gamma \left(\sum k_\gamma \psi_\gamma \right) := \sum \bar{k}_\gamma \psi_\gamma, k_\gamma \in \mathbb{C}; \quad J_\Gamma^* = J_\Gamma; \quad J_\Gamma^2 = I. \quad (22)$$

Note that for $A \in L^1(\mathcal{L})$ we have:

$$\begin{aligned} \text{Tr}_{\mathcal{L}}(J_\Gamma A J_\Gamma) &= \sum (\psi_\gamma | J_\Gamma A J_\Gamma \psi_\gamma) = \sum (A J_\Gamma \psi_\gamma | J_\Gamma \psi_\gamma) = \sum (A \psi_\gamma | \psi_\gamma) = \\ &= \text{Tr}_{\mathcal{L}}(A^*) \end{aligned} \quad (23)$$

In the same way for a basis $\Omega = \{\phi_\omega\}$ in \mathcal{R} we have the mapping $\tilde{\Omega}$

$$\tilde{\Omega} : L^2(\mathcal{R}, \mathcal{L}) \ni d \mapsto \tilde{\Omega}(d) = \sum d \phi_\omega \otimes \phi_\omega \in \mathcal{L} \otimes \mathcal{R};$$

with the corresponding antiunitary involution $J_\Omega : J_\Omega \left(\sum l_\omega \phi_\omega \right) := \sum \bar{l}_\omega \phi_\omega$, and

Lemma 2.6 *The map $\tilde{\Omega}$ is unitary, moreover for $A \in \mathcal{B}(\mathcal{L})$, $B \in L^1(\mathcal{R})$ and $b, d \in L^2(\mathcal{R}, \mathcal{L})$:*

$$\begin{aligned} \tilde{\Omega}(Ad) &= (A \otimes I) \tilde{\Omega}(d) \\ (\tilde{\Omega}(d) | (A \otimes I) \tilde{\Omega}(b)) &= (\tilde{\Omega}(d) | \tilde{\Omega}(Ab)) = \text{Tr}_{\mathcal{R}}(d^* Ab) = \text{Tr}_{\mathcal{L}}(Abd^*) = \text{Tr}_{\mathcal{L}}(bd^* A) \\ \text{Tr}_{\mathcal{R}}(J_\Omega B J_\Omega) &= \text{Tr}_{\mathcal{R}}(B^*) \end{aligned} \quad (24)$$

The choice of bases in \mathcal{L} and \mathcal{R} gives us all of these objects and the following straightforward lemma describes relations between both structures:

Lemma 2.7 *Let $c \in L^2(\mathcal{L}, \mathcal{R})$ and $d \in L^2(\mathcal{R}, \mathcal{L})$. Then the following equalities hold:*

$$\tilde{\Gamma}(c) = \tilde{\Omega}(J_\Gamma c^* J_\Omega), \quad \tilde{\Gamma}(J_\Omega d^* J_\Gamma) = \tilde{\Omega}(d) \quad (25)$$

$$\begin{aligned} (\tilde{\Omega}(d) | \tilde{\Gamma}(c)) &= (\tilde{\Omega}(d) | \tilde{\Omega}(J_\Gamma c^* J_\Omega)) = \text{Tr}_{\mathcal{R}}(d^* J_\Gamma c^* J_\Omega) = \\ &= (\tilde{\Gamma}(J_\Omega d^* J_\Gamma) | \tilde{\Gamma}(c)) = \text{Tr}_{\mathcal{L}}(J_\Gamma d J_\Omega c) \end{aligned} \quad (26)$$

Clearly, the choice of bases Γ, Ω is equivalent to the choice of a basis Γ and a unitary operator $U : \mathcal{L} \rightarrow \mathcal{R}$ (the equality $U(\psi_\gamma) = \phi_\gamma$ is a definition of operator U or a basis $\Omega = \{\phi_\gamma\}$). It is sometimes more convenient to use pair $(\tilde{\Gamma}, U)$ instead of $(\tilde{\Gamma}, \tilde{\Omega})$. In the lemma below we collect formulae we will use:

Lemma 2.8 *For $c \in L^2(\mathcal{L}, \mathcal{R})$, $A \in \mathcal{B}(\mathcal{L})$, $B \in \mathcal{B}(\mathcal{R})$ the following equalities hold:*

$$J_\Omega = U J_\Gamma U^* \quad (27)$$

$$(\tilde{\Gamma}(c) | (I \otimes B) \tilde{\Gamma}(c)) = (\tilde{\Gamma}(|c^*|U) | (I \otimes B) \tilde{\Gamma}(|c^*|U)) \quad (28)$$

$$(\tilde{\Gamma}(c) | (A \otimes I) \tilde{\Gamma}(c)) = (\tilde{\Gamma}(U|c|) | (A \otimes I) \tilde{\Gamma}(U|c|)) \quad (29)$$

$$(\tilde{\Gamma}(c) | (A \otimes I) \tilde{\Gamma}(c)) = (\tilde{\Gamma}(U|c|) | (I \otimes U J_\Gamma A^* J_\Gamma U^*) \tilde{\Gamma}(U|c|)) \quad (30)$$

$$(\tilde{\Gamma}(c) | (I \otimes B) \tilde{\Gamma}(c)) = (\tilde{\Gamma}(|c^*|U) | (J_\Gamma U^* B^* U J_\Gamma \otimes I) \tilde{\Gamma}(|c^*|U)) \quad (31)$$

$$(\tilde{\Gamma}(c) | (A \otimes B) \tilde{\Gamma}(c)) = \text{Tr}_{\mathcal{L}}(c^* B c J_\Gamma A^* J_\Gamma) \quad (32)$$

Proof: For $c \in L^2(\mathcal{L}, \mathcal{R})$ recall that $|c| := \sqrt{c^* c} \in L^2(\mathcal{L})$ and $|c^*| = \sqrt{c c^*} \in L^2(\mathcal{R})$; notice also that $U|c| \in L^2(\mathcal{L}, \mathcal{R})$ and $|c^*|U \in L^2(\mathcal{L}, \mathcal{R})$.

The formula (27) is straightforward, let us prove (28). For $B \in \mathcal{B}(\mathcal{R})$, compute:

$$\begin{aligned} (\tilde{\Gamma}(c) | (I \otimes B) \tilde{\Gamma}(c)) &= \text{Tr}_{\mathcal{L}}(c^* B c) = \text{Tr}_{\mathcal{R}}(c c^* B) = \text{Tr}_{\mathcal{R}}(|c^*|^2 B) = \text{Tr}_{\mathcal{R}}(|c^*| B |c^*|) = \\ &= \text{Tr}_{\mathcal{L}}(U^* |c^*| B |c^*| U) = \text{Tr}_{\mathcal{L}}((|c^*| U)^* B |c^*| U) = \\ &= (\tilde{\Gamma}(|c^*|U) | (I \otimes B) \tilde{\Gamma}(|c^*|U)) \end{aligned}$$

For the next formula, let $A \in \mathcal{B}(\mathcal{L})$ and using lemma 2.7 we compute:

$$\begin{aligned} \left(\tilde{\Gamma}(c) \mid (A \otimes I) \tilde{\Gamma}(c) \right) &= \left(\tilde{\Omega}(J_{\Gamma} c^* J_{\Omega}) \mid (A \otimes I) \tilde{\Omega}(J_{\Gamma} c^* J_{\Omega}) \right) = \text{Tr}_{\mathcal{R}}((J_{\Gamma} c^* J_{\Omega})^* A J_{\Gamma} c^* J_{\Omega}) = \\ &= \text{Tr}_{\mathcal{R}}(J_{\Omega} c J_{\Gamma} A (J_{\Gamma} c^* J_{\Omega})) = \text{Tr}_{\mathcal{L}}(J_{\Gamma} c^* J_{\Omega} J_{\Omega} c J_{\Gamma} A) = \\ &= \text{Tr}_{\mathcal{L}}(J_{\Gamma} |c|^2 J_{\Gamma} A) \end{aligned}$$

Writing the formula above for $U|c|$ instead of c and noting that $|U|c|^2 = (U|c|)^* U|c| = |c|^2$ we obtain the equality (29).

We prove the equality (30):

$$\begin{aligned} \left(\tilde{\Gamma}(c) \mid (A \otimes I) \tilde{\Gamma}(c) \right) &= \text{Tr}_{\mathcal{L}}(J_{\Gamma} |c|^2 J_{\Gamma} A) = \text{Tr}_{\mathcal{L}}((J_{\Gamma} |c| J_{\Gamma}) J_{\Gamma} |c| J_{\Gamma} A) = \text{Tr}_{\mathcal{L}}(J_{\Gamma} |c| J_{\Gamma} A J_{\Gamma} |c| J_{\Gamma}) = \\ &= \text{Tr}_{\mathcal{L}}(|c| J_{\Gamma} A^* J_{\Gamma} |c|) = \text{Tr}_{\mathcal{L}}((U|c|)^* U J_{\Gamma} A^* J_{\Gamma} U (U|c|)) = \\ &= \left(\tilde{\Gamma}(U|c|) \mid (I \otimes U J_{\Gamma} A^* J_{\Gamma} U^*) \tilde{\Gamma}(U|c|) \right) \end{aligned}$$

and the formula (31):

$$\begin{aligned} \left(\tilde{\Gamma}(c) \mid (I \otimes B) \tilde{\Gamma}(c) \right) &= \text{Tr}_{\mathcal{R}}(|c^*| B |c^*|) = \text{Tr}_{\mathcal{R}}(J_{\Omega} |c^*| B^* |c^*| J_{\Omega}) = \\ &= \text{Tr}_{\mathcal{R}}(J_{\Omega} |c^*| U J_{\Gamma} J_{\Gamma} U^* B^* U J_{\Gamma} J_{\Gamma} U^* |c^*| J_{\Omega}) = \\ &= \text{Tr}_{\mathcal{R}}((J_{\Gamma} U^* |c^*| J_{\Omega})^* (J_{\Gamma} U^* B^* U J_{\Gamma}) (J_{\Gamma} U^* |c^*| J_{\Omega})) = \\ &= \left(\tilde{\Omega}(J_{\Gamma} U^* |c^*| J_{\Omega}) \mid (J_{\Gamma} U^* B^* U J_{\Gamma} \otimes I) \tilde{\Omega}(J_{\Gamma} U^* |c^*| J_{\Omega}) \right) = \\ &= \left(\tilde{\Gamma}(|c^*| U) \mid (J_{\Gamma} U^* B^* U J_{\Gamma} \otimes I) \tilde{\Gamma}(|c^*| U) \right) \end{aligned}$$

Finally, we prove (32):

$$\begin{aligned} \left(\tilde{\Gamma}(c) \mid (A \otimes B) \tilde{\Gamma}(c) \right) &= \left((A^* \otimes I) \tilde{\Gamma}(c) \mid (I \otimes B) \tilde{\Gamma}(c) \right) = \left((A^* \otimes I) \tilde{\Omega}(J_{\Gamma} c^* J_{\Omega}) \mid \tilde{\Gamma}(Bc) \right) = \\ &= \left(\tilde{\Omega}(A^* J_{\Gamma} c^* J_{\Omega}) \mid \tilde{\Gamma}(Bc) \right) = \text{Tr}_{\mathcal{R}}(J_{\Omega} c J_{\Gamma} A J_{\Gamma} c^* B^* J_{\Omega}) = \\ &= \text{Tr}_{\mathcal{R}}(Bc J_{\Gamma} A^* J_{\Gamma} c^*) = \text{Tr}_{\mathcal{L}}(J_{\Gamma} A^* J_{\Gamma} c^* J_{\Omega} J_{\Omega} Bc) = \\ &= \text{Tr}_{\mathcal{L}}(J_{\Gamma} A^* J_{\Gamma} c^* Bc) = \text{Tr}_{\mathcal{L}}(c^* Bc J_{\Gamma} A^* J_{\Gamma}), \end{aligned}$$

where we have used (26). ■

Now we want to express the inequality (9) i.e.

$$2|\text{Tr}_{\mathcal{L}}(c^* Bc A^*)| \leq \text{Tr}_{\mathcal{L}}(|c| A |c| A^*) + \text{Tr}_{\mathcal{R}}(|c^*| B |c^*| B^*) \quad (33)$$

in terms of $\tilde{\Gamma}, J_{\Gamma}, U$ and the scalar product in $\mathcal{L} \otimes \mathcal{R}$.

In the formula (32) we put $B := U J_{\Gamma} A J_{\Gamma} U^*$ and $U|c|$ instead of c and get:

$$\begin{aligned} \left(\tilde{\Gamma}(U|c|) \mid (A \otimes (U J_{\Gamma} A J_{\Gamma} U^*)) \tilde{\Gamma}(U|c|) \right) &= \text{Tr}_{\mathcal{L}}(|c| U^* (U J_{\Gamma} A J_{\Gamma} U^*) (U|c|) J_{\Gamma} A^* J_{\Gamma}) = \\ &= \text{Tr}_{\mathcal{L}}(|c| J_{\Gamma} A J_{\Gamma} |c| J_{\Gamma} A^* J_{\Gamma}) \end{aligned}$$

Now put into (32) $A = U^* J_{\Omega} B J_{\Omega} U$ and $|c^*| U$ instead of c and obtain:

$$\begin{aligned} \left(\tilde{\Gamma}(|c^*| U) \mid (U^* J_{\Omega} B J_{\Omega} U \otimes B) \tilde{\Gamma}(|c^*| U) \right) &= \text{Tr}_{\mathcal{L}}(U^* |c^*| B |c^*| U J_{\Gamma} (U^* J_{\Omega} B J_{\Omega} U)^* J_{\Gamma}) = \\ &= \text{Tr}_{\mathcal{L}}(U^* |c^*| B |c^*| (U J_{\Gamma} U^*) J_{\Omega} B^* J_{\Omega} U J_{\Gamma}) = \text{Tr}_{\mathcal{L}}(U^* |c^*| B |c^*| (J_{\Omega} J_{\Omega}) B^* J_{\Omega} (J_{\Omega} U)) = \\ &= \text{Tr}_{\mathcal{L}}(U^* |c^*| B |c^*| B^* U) = \\ &= \text{Tr}_{\mathcal{R}}(|c^*| B |c^*| B^*), \end{aligned}$$

where (27) i.e. $J_{\Omega} = U J_{\Gamma} U^*$ was used.

Finally, writing (33) with $J_{\Gamma} A J_{\Gamma}$ instead of A , using (32) and two equalities above we obtain:

Proposition 2.9 *Let $A \in \mathcal{B}(\mathcal{L})$, $B \in \mathcal{B}(\mathcal{R})$ and $c \in L^2(\mathcal{L}, \mathcal{R})$. Then*

$$2 \left| \left(\tilde{\Gamma}(c) \mid (A \otimes B) \tilde{\Gamma}(c) \right) \right| \leq \left(\tilde{\Gamma}(U|c|) \mid (A \otimes U J_{\Gamma} A J_{\Gamma} U^*) \tilde{\Gamma}(U|c|) \right) + \left(\tilde{\Gamma}(|c^*|U) \mid (U^* J_{\Omega} B J_{\Omega} U \otimes B) \tilde{\Gamma}(|c^*|U) \right) \quad (34)$$

We will also need formulae similar to the ones in Lemma 2.8 for some unbounded operators, they are proven in the following proposition.

Proposition 2.10 *Let T and S be self-adjoint operators with purely point spectrum acting on \mathcal{L} and \mathcal{R} respectively. Assume that bases $\{\psi_{\gamma}\}$ and $\{\phi_{\gamma}\}$ consist of eigenvectors of T and S : $T\psi_{\gamma} = t_{\gamma}\psi_{\gamma}$ and $S\phi_{\gamma} = s_{\gamma}\phi_{\gamma}$; assume moreover that $\tilde{\Gamma}(c) \in D((T \otimes I)^2)$ and $\tilde{\Omega}(d) \in D((I \otimes S)^2)$. Then:*

$$\left(\tilde{\Gamma}(c) \mid (T \otimes I) \tilde{\Gamma}(c) \right) = \left(\tilde{\Gamma}(U|c|) \mid (T \otimes I) \tilde{\Gamma}(U|c|) \right) = \left(\tilde{\Gamma}(U|c|) \mid (I \otimes U T U^*) \tilde{\Gamma}(U|c|) \right) \quad (35)$$

$$\left(\tilde{\Omega}(d) \mid (I \otimes S) \tilde{\Omega}(d) \right) = \left(\tilde{\Omega}(U^*|d|) \mid (I \otimes S) \tilde{\Omega}(U^*|d|) \right) = \left(\tilde{\Omega}(U^*|d|) \mid (U^* S U \otimes I) \tilde{\Omega}(U^*|d|) \right) \quad (36)$$

Proof: We will prove (35); equalities in (36) can be proven in a similar manner. Since $\tilde{\Gamma}(c)$ is in the domain of $(T \otimes I)^2$ we have:

$$\begin{aligned} \left(\tilde{\Gamma}(c) \mid (T \otimes I)^2 \tilde{\Gamma}(c) \right) &= \sum_{\gamma} (\psi_{\gamma} \otimes c\psi_{\gamma} \mid (T \otimes I)^2 \tilde{\Gamma}(c)) = \sum_{\gamma} (t_{\gamma}^2 \psi_{\gamma} \otimes c\psi_{\gamma} \mid \tilde{\Gamma}(c)) = \\ &= \sum_{\gamma} |t_{\gamma}|^2 \|c\psi_{\gamma}\|^2. \end{aligned} \quad (37)$$

This equality means that the series $\sum_{\gamma} t_{\gamma} \psi_{\gamma} \otimes c\psi_{\gamma}$ is convergent. Because $T \otimes I$ is closed and $\psi_{\gamma} \otimes c\psi_{\gamma} \in D(T \otimes I)$ it implies that:

$$(T \otimes I) \tilde{\Gamma}(c) = \sum_{\gamma} t_{\gamma} \psi_{\gamma} \otimes c\psi_{\gamma} \quad (38)$$

Since $\|U|c|\psi_{\gamma}\|^2 = \|c\psi_{\gamma}\|^2$ the formula (37) implies also convergence of the series $\sum_{\gamma} t_{\gamma} \psi_{\gamma} \otimes U|c|\psi_{\gamma}$ and the equality

$$(T \otimes I) \tilde{\Gamma}(U|c|) = \sum_{\gamma} t_{\gamma} \psi_{\gamma} \otimes U|c|\psi_{\gamma} \quad (39)$$

By the lemma 2.7: $\tilde{\Gamma}(U|c|) = \tilde{\Omega}(J_{\Gamma}|c|U^*J_{\Omega})$. Since $J_{\Gamma}|c|U^*\phi_{\gamma} \otimes \phi_{\gamma} = J_{\Gamma}|c|\psi_{\gamma} \otimes \phi_{\gamma} \in D(I \otimes U T U^*)$ the formula (37) means convergence of the series $\sum_{\gamma} t_{\gamma} J_{\Gamma}|c|\psi_{\gamma} \otimes \phi_{\gamma}$ and the equality

$$(I \otimes U T U^*) \tilde{\Gamma}(U|c|) = \sum_{\gamma} t_{\gamma} (J_{\Gamma}|c|\psi_{\gamma}) \otimes \phi_{\gamma} \quad (40)$$

follows. Now combining (38), (39), (40) and $\tilde{\Gamma}(U|c|) = \tilde{\Omega}(J_{\Gamma}|c|U^*J_{\Omega})$ we obtain (35). ■

3 Ground state ordering in the system of 2-dimensional rotors

3.1 Description of the system

Denote by Λ the finite subset of the simple cubic lattice in d dimensions: $\Lambda \subset \mathbb{Z}^d$. We assume that Λ is a (discrete) hypercube and that *the number of sites along every edge is even*; let us fix $2N$ to be the length of the hypercube edge:

$$\Lambda := \{\mathbf{x} \in \mathbb{Z}^d : -N + 1 \leq x_i \leq N, i = 1, \dots, d\} \quad (41)$$

With every site $\mathbf{x} \in \Lambda$ we associate a real variable $\varphi_{\mathbf{x}} \in [0, 2\pi[$. In physical terms, it describes the (angular) position of the rotor at the site \mathbf{x} . Equivalently, the position of the rotor at the site \mathbf{x} can be described as a unit vector $\mathbf{s}_{\mathbf{x}} \in \mathbb{S}^1$, i.e. one dimensional torus \mathbb{T} :

$$\mathbf{s}_{\mathbf{x}} = [s_{\mathbf{x}}^x, s_{\mathbf{x}}^y] = [\cos \varphi_{\mathbf{x}}, \sin \varphi_{\mathbf{x}}].$$

The total spin is $\mathbf{S} = \sum_{\mathbf{x} \in \Lambda} \mathbf{s}_{\mathbf{x}}$. The Hilbert space $\mathcal{H}_{\mathbf{x}}$ of states on a given site \mathbf{x} is the space of square integrable periodic functions, i.e. $\mathcal{H}_{\mathbf{x}} = L^2(\mathbb{T})$. The Hilbert space associated to the whole system is the space of square integrable functions on $|\Lambda|$ - dimensional torus:

$$\mathcal{H}_{\Lambda} = L^2(\mathbb{T}^{|\Lambda|}) = \bigotimes_{\mathbf{x} \in \Lambda} \mathcal{H}_{\mathbf{x}}$$

The operator T of total kinetic energy of the system of rotors is proportional to the laplacian Δ :

$$T = -\frac{1}{2I} \sum_{\mathbf{x} \in \Lambda} \frac{\partial^2}{\partial \varphi_{\mathbf{x}}^2}, \quad (42)$$

where $I > 0$ is the moment of inertia of rotor (we assume that all rotors have equal moments of inertia). The system of interacting rotors is defined by the Hamiltonian $H = T + \hat{V}$, where \hat{V} is an interaction energy between rotors; it is an operator of multiplication by a smooth function V . We shall consider the Hamiltonian:

$$H = T + \hat{V}_0, \quad V_0 := -J \sum_{\langle \mathbf{x}, \mathbf{y} \rangle} \cos(\varphi_{\mathbf{x}} - \varphi_{\mathbf{y}}) \quad (43)$$

In this formula $\langle \mathbf{x}, \mathbf{y} \rangle$ means that \mathbf{x} and \mathbf{y} are the nearest neighbours. By this we mean that all but one coordinates of \mathbf{x} and \mathbf{y} are equal and the ones, say x_i and y_i , that differ satisfy $|x_i - y_i| = 1 \bmod 2(N-1)$. Thus a site \mathbf{x} laying on the hyperplane defined by $x_i = N$ has some of its nearest neighbours on a hyperplanes defined by $x_i = -N + 1$.

J is the coupling constant: $J > 0$ corresponds to ferromagnetic coupling between rotors and $J < 0$ to the antiferromagnetic one. In what follows we shall restrict ourselves to the ferromagnetic case, as only in this situation the Reflection Positivity arguments may be applied.

Our Hamiltonian (43) is an elliptic second order differential operator on a $|\Lambda|$ -dimensional torus (compact manifold). It is a special case of the more general situation:

Theorem 3.1 [23] *Let (M, g) be a compact, oriented, riemannian smooth manifold without boundary; $L : C^\infty(M) \rightarrow C^\infty(M)$ formally selfadjoint, linear, elliptic, PDO of order $k > 0$. Then:*

1. L extends uniquely to $\tilde{L} : H^k(M) \rightarrow L^2(M)$; ($H^k(M)$ is k -th Sobolev space)
2. \tilde{L} as an operator on $L^2(M)$ (with the domain $H^k(M)$) is selfadjoint;
3. The spectrum of \tilde{L} consists of isolated eigenvalues of finite multiplicity;
4. Eigenvectors of \tilde{L} are smooth functions.

Let us observe that since T is positive and V_0 is a continuous function, the hamiltonian H is bounded from below.

In the following we need in an essential way the uniqueness of the ground state of the Hamiltonian. This is a consequence of positivity improving property of the semigroup $\exp(-t\Delta)$. For convenience of the reader, we recall briefly main definitions and results (we refer to Chapt. XIII of [16] for a detailed presentation).

A non zero function $\Psi \in L^2(M)$ is positive iff $\Psi(x) \geq 0$; it is strictly positive if $\Psi(x) > 0$ (both inequalities should be understood in almost everywhere sense).

A bounded operator A is:

- positivity preserving if $A\Psi$ is positive for positive Ψ ;
- positivity improving if $A\Psi$ is strictly positive for positive Ψ ; equivalent condition is that $(\Psi | A\Phi) > 0$ for positive Ψ and Φ .

The following is, simplified for our needs, Thm XIII.44 from [16].

Proposition 3.2 *Let H be a self adjoint, bounded from below operator on $L^2(M)$. Assume the spectrum of H consists of isolated eigenvalues of finite multiplicity. If for every $t > 0$ the operator $\exp(-tH)$ is positivity improving then the ground state of H is unique (and strictly positive).*

In our situation $M = \mathbb{T}^{|\Lambda|}$, and it is known that for kinetic energy operator T given by the formula (42) operators $\exp(-tT)$, $t > 0$ are integral operators; for $\Phi, \Psi \in L^2(\mathbb{T}^{|\Lambda|})$:

$$(\Phi | \exp(-tT)\Psi) = \int dx dy \overline{\Phi(x)} K(t, x, y) \Psi(y).$$

The function $K(t, x, y)$, $t > 0, x, y \in \mathbb{T}^{|\Lambda|}$ - the heat kernel for $|\Lambda|$ - torus, is explicitly known; it is *a strictly positive* function, therefore $\exp(-tT)$ is positivity improving.

The same property for $H = T + \hat{V}$ holds due to the Trotter product formula (see e.g Chapt. X of [15]):

$$\exp(-t(T + \hat{V}))\Psi = \lim_{n \rightarrow \infty} \left[\exp(-tT/n) \exp(-t\hat{V}/n) \right]^n \Psi$$

In fact, for any $\Phi, \Psi \in \mathcal{H}_\Lambda$:

$$(\Phi | \exp(-tH)\Psi) = \lim_{n \rightarrow \infty} \left(\Phi | \left[\exp(-tT/n) \exp(-t\hat{V}/n) \right]^n \Psi \right)$$

Let $C := \sup V$. Since $\exp(-tT)$ is positivity improving, for strictly positive numbers: $\alpha, \beta, \gamma, \delta > 0$ and positive functions Φ and Ψ we have:

$$\begin{aligned} \left(\Phi | \exp(-\alpha T) \exp(-\beta \hat{V}) \exp(-\gamma T) \exp(-\delta \hat{V}) \Psi \right) &\geq e^{-\beta C} \left(\exp(-\alpha T) \Phi | \exp(-\gamma T) \exp(-\delta \hat{V}) \Psi \right) \geq \\ &\geq e^{-(\beta+\delta)C} (\Phi | \exp(-(\alpha + \gamma)T) \Psi). \end{aligned}$$

In particular $\left(\Phi | \left[\exp(-tT/n) \exp(-t\hat{V}/n) \right]^n \Psi \right) \geq e^{-tC} (\Phi | \exp(-tT) \Psi) > 0$.

Therefore the inequality is preserved in the limit: $(\Phi | \exp(-tH)\Psi) > 0$ and the uniqueness of the ground state of H follows.

3.2 Criteria of ordering

The simplest definition of the order parameter would be an average of the total spin. However, this definition is of little use for the zero field (i.e. as a measure of the spontaneous magnetization) as it is zero due to symmetry. The more physical definition is a zero-field limit of magnetization: $\mathcal{M} = \lim_{h \rightarrow 0} M(h)$ (h denotes magnetic field) but it is difficult to deal with. More easy to handle is the *average of the square of spin*. It follows that if the average of the square of spin is different from zero, then the zero-field magnetization is non-zero, too (Griffiths theorem – see [1]). So, we take the average $\langle \mathbf{S}^2 \rangle$ as a measure of order parameter. *All averages considered in this paper are taken over the ground state.* Following this idea, we will prove that if I and J are sufficiently large the ground state of system of interacting rotors, described by the potential (43), exhibits Long-Range Order (LRO):

Theorem 3.3 *Assume that I and J satisfy the inequality*

$$\sqrt{IJ} > \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \frac{d\mathbf{k}}{\sqrt{\mathcal{E}(\mathbf{k})}} \equiv \mathcal{I}_d, \quad (44)$$

where the function $\mathcal{E} : \mathbb{R}^d \rightarrow \mathbb{R}$ is defined by:

$$\mathcal{E}(\mathbf{k}) = d - \sum_{i=1}^d \cos k_i. \quad (45)$$

Then there exists $C > 0$ such that, for sufficiently large $|\Lambda|$:

$$\left\langle \left(\frac{\mathbf{S}^x}{|\Lambda|} \right)^2 \right\rangle \equiv \left\langle \left(\frac{1}{|\Lambda|} \sum_{\mathbf{x} \in \Lambda} s_{\mathbf{x}}^x \right)^2 \right\rangle \equiv \left\langle \left(\frac{1}{|\Lambda|} \sum_{\mathbf{x} \in \Lambda} \cos \varphi_{\mathbf{x}} \right)^2 \right\rangle \geq C, \quad (46)$$

Remarks.

1. The estimation of the form similar to the one in (46) appeared in P-based proofs of LRO in other classes of models including classical and quantum spin systems [1], [4], [5].
2. The integral \mathcal{I}_1 is divergent, so above theorem does not prove existence of ground-state LRO in $d = 1$ but scaling and field-theoretical arguments [12] show (non-rigorously) that there is no ordering in this case.
3. For $d \geq 2$ the integral \mathcal{I}_d is finite, so there is LRO in the ground-state. Numerical values of \mathcal{I}_d for physical dimensions are: $\mathcal{I}_2 \approx 0.909173$; $\mathcal{I}_3 \approx 0.643954$. It indicates the tendency to ordering increases with the growth of a dimension.

4. For positive temperatures, there is no ordering in dimensions 1 and 2. This statement, analogous to the famous Mermin-Wagner theorem for spin systems [13], has been proven in [5].
5. It is known that in $d \geq 3$, the LRO is present in sufficiently low temperatures, and so in the ground state, too [5]. Therefore the Theorem 3.3 is most interesting in $d = 2$. Such a result, using another RP arguments, has been proven in [10]. In different approach, by scaling and field-theoretical arguments, it was obtained (non-rigorously) in [12].

3.3 Estimations for basic functions

To prove the theorem 3.3 it will be convenient to work with Fourier-transformed spins. Let us define:

$$\hat{s}_{\mathbf{k}}^\alpha = \frac{1}{\sqrt{|\Lambda|}} \sum_{\mathbf{x} \in \Lambda} s_{\mathbf{x}}^\alpha e^{i\mathbf{k} \cdot \mathbf{x}} \quad (\alpha = x, y) \quad (47)$$

where \mathbf{k} takes value in the first Brillouin zone, i.e. $k_j \in \{-\frac{\pi(N-1)}{N}, \dots, \frac{\pi(N-1)}{N}, \pi\}$ for $j = 1, \dots, d$. Let us remark that due to the symmetry of our system we have:

Lemma 3.4 *For the system of rotors described by the hamiltonian (43) operators $\hat{s}_{\mathbf{k}}^\alpha$ ($\alpha = x, y$) satisfy identities:*

$$\langle \hat{s}_{\mathbf{k}}^\alpha \rangle = 0 \quad \text{and} \quad \langle \hat{s}_{\mathbf{k}}^x (\hat{s}_{\mathbf{k}}^x)^* \rangle = \langle \hat{s}_{\mathbf{k}}^y (\hat{s}_{\mathbf{k}}^y)^* \rangle. \quad (48)$$

Proof: Let R_θ be the rotation (in all variables) by θ :

$$(R_\theta f)(\varphi_1, \dots, \varphi_{|\Lambda|}) := f(\varphi_1 - \theta, \dots, \varphi_{|\Lambda|} - \theta)$$

It is clear that R_θ is unitary. Since the hamiltonian (43) commutes with R_θ and the ground state is unique we have $R_\theta \psi_0 = \lambda \psi_0$ for a complex number λ with $|\lambda| = 1$. Therefore

$$\langle A \rangle = (\psi_0 | A \psi_0) = (R_\theta \psi_0 | A R_\theta \psi_0) = \langle R_\theta^* A R_\theta \rangle$$

It is easy to check that for any \mathbf{k} : $R_\pi \hat{s}_{\mathbf{k}}^\alpha = -\hat{s}_{\mathbf{k}}^\alpha R_\pi$ and $R_{\pi/2} \hat{s}_{\mathbf{k}}^x = \hat{s}_{\mathbf{k}}^y R_{\pi/2}$. Now equalities (48) are clear. \blacksquare

Let us denote by $g_{\mathbf{k}}$ the two-point correlation function in the momentum representation:

$$g_{\mathbf{k}} = \langle \hat{s}_{\mathbf{k}}^x (\hat{s}_{\mathbf{k}}^x)^* \rangle \quad (49)$$

Clearly $g_{\mathbf{k}} \geq 0$. Since $\hat{s}_{\mathbf{k}}^x$ is normal $g_{\mathbf{k}} = \langle (\hat{s}_{\mathbf{k}}^x)^* \hat{s}_{\mathbf{k}}^x \rangle$. Remember also that $g_{\mathbf{k}}$ depends on parameters I and J . The $g_{\mathbf{k}}$ function and its estimation will play a crucial role in the proof of existence of spontaneous magnetization.

With this notation the inequality (46) can be rewritten as

$$\frac{1}{|\Lambda|} g_0 \geq C > 0. \quad (50)$$

The strategy of the proof of Theorem 3.3 can be described as follows (general ideas are similar to ones in [4], [1], [2]). First (and rather easy) step is the equality

$$\sum_{\mathbf{k}} g_{\mathbf{k}} = \frac{1}{2} |\Lambda| \quad (51)$$

Indeed, for the ground state ψ_0 of H :

$$\sum_{\mathbf{k}} g_{\mathbf{k}} = \sum_{\mathbf{k}} (\hat{s}_{\mathbf{k}}^x \psi_0 | \hat{s}_{\mathbf{k}}^x \psi_0) = \sum_{\mathbf{k}, \mathbf{x}, \mathbf{y}} \frac{1}{|\Lambda|} (\cos \varphi_{\mathbf{x}} e^{i\mathbf{k} \cdot \mathbf{x}} \psi_0 | \cos \varphi_{\mathbf{y}} e^{i\mathbf{k} \cdot \mathbf{y}} \psi_0) = \sum_{\mathbf{x}} (\cos^2 \varphi_{\mathbf{x}} \psi_0 | \psi_0),$$

where we have used the obvious formula: $\sum_{\mathbf{k}} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})} = |\Lambda| \delta(\mathbf{x} - \mathbf{y})$. In the same way we obtain:

$\sum_{\mathbf{k}} (\hat{s}_{\mathbf{k}}^y \psi_0 | \hat{s}_{\mathbf{k}}^y \psi_0) = \sum_{\mathbf{x}} (\sin^2 \varphi_{\mathbf{x}} \psi_0 | \psi_0)$. Adding these two equalities we get:

$$\sum_{\mathbf{k}} \langle \hat{s}_{\mathbf{k}}^x (\hat{s}_{\mathbf{k}}^x)^* \rangle + \langle \hat{s}_{\mathbf{k}}^y (\hat{s}_{\mathbf{k}}^y)^* \rangle = |\Lambda|$$

and the equality (51) follows due to the second formula of (48)

It turns out, that for $\mathbf{k} \neq \mathbf{0}$, and this is the place where the RP arguments and the main inequality (34) is used, the function $g_{\mathbf{k}}$ can be estimated from above by an integrable (for $d > 1$) function:

$$g_{\mathbf{k}} \leq \frac{1}{2\sqrt{IJ}\sqrt{\mathcal{E}(\mathbf{k})}}, \quad \mathbf{k} \neq \mathbf{0}. \quad (52)$$

where the function $\mathcal{E}(\mathbf{k})$ was defined in (45).

With (51) and (52) in hand we can write

$$\frac{1}{|\Lambda|} g_0 = \frac{1}{2} - \sum_{\mathbf{k} \neq 0} g_{\mathbf{k}} \geq \frac{1}{2\sqrt{IJ}} \left(\sqrt{IJ} - \frac{1}{|\Lambda|} \sum_{\mathbf{k} \neq 0} \frac{1}{\sqrt{\mathcal{E}(\mathbf{k})}} \right)$$

Now the sum $\frac{(2\pi)^d}{|\Lambda|} \sum_{\mathbf{k} \neq 0} \frac{1}{\sqrt{\mathcal{E}(\mathbf{k})}}$ converges as $\Lambda \rightarrow \mathbb{Z}^d$ to the integral $\int_{[-\pi, \pi]^d} \frac{d\mathbf{k}}{\sqrt{\mathcal{E}(\mathbf{k})}}$. Therefore the inequality (44) implies that there exists $C > 0$ such that for sufficiently large Λ we have

$$\left(\sqrt{IJ} - \frac{1}{|\Lambda|} \sum_{\mathbf{k} \neq 0} \frac{1}{\sqrt{\mathcal{E}(\mathbf{k})}} \right) \geq C$$

i.e the estimate (46). This way to complete the proof of the theorem 3.3 it remains to prove the estimate (52). The existence of such an estimate seems to be a quite general phenomenon, but at present we can prove it only using RP techniques.

The inequality (46) can be viewed as the appearance of the macroscopic occupation of the $\mathbf{k} = \mathbf{0}$ mode. In the other words, it is an indication that in the thermodynamic limit, the $g_{\mathbf{k}}$ function possess non-zero δ function contribution at $\mathbf{k} = 0$.

We will get the inequality (52) by relating $g_{\mathbf{k}}$ to other functions, in particular *susceptibility* (57), for which we will get an estimate by RP techniques.

Let $(\psi_n)_{n \geq 0}$ be an orthonormal basis consisting of eigenvectors of the Hamiltonian (43) with $H\psi_n = E_n\psi_n$ and ψ_0 be the ground state. We have:

$$\begin{aligned} g_{\mathbf{k}} &= \langle (\hat{s}_{\mathbf{k}}^x)^* \hat{s}_{\mathbf{k}}^x \rangle = (\hat{s}_{\mathbf{k}}^x \psi_0 | \hat{s}_{\mathbf{k}}^x \psi_0) = \sum_{n \geq 0} (\hat{s}_{\mathbf{k}}^x \psi_0 | \psi_n) (\psi_n | \hat{s}_{\mathbf{k}}^x \psi_0) = \\ &= \sum_{n > 0} (\hat{s}_{\mathbf{k}}^x \psi_0 | \psi_n) (\psi_n | \hat{s}_{\mathbf{k}}^x \psi_0) = \sum_{n > 0} |(\hat{s}_{\mathbf{k}}^x \psi_0 | \psi_n)|^2. \end{aligned} \quad (53)$$

There is no term with $n = 0$ due to the first formula of (48). Since $\hat{s}_{\mathbf{k}}^x$ is normal $g_{\mathbf{k}}$ can be writtes as:

$$g_{\mathbf{k}} = \langle \hat{s}_{\mathbf{k}}^x (\hat{s}_{\mathbf{k}}^x)^* \rangle = \sum_{n > 0} |((\hat{s}_{\mathbf{k}}^x)^* \psi_0 | \psi_n)|^2 = \frac{1}{2} \sum_{n > 0} \left[|(\hat{s}_{\mathbf{k}}^x \psi_0 | \psi_n)|^2 + |((\hat{s}_{\mathbf{k}}^x)^* \psi_0 | \psi_n)|^2 \right] \quad (54)$$

Next, we define:

$$\mathcal{D}_{\mathbf{k}} := \frac{1}{2} \langle [[\hat{s}_{\mathbf{k}}^x, H], (\hat{s}_{\mathbf{k}}^x)^*] \rangle \quad (55)$$

Since $(\hat{s}_{\mathbf{k}}^x)\psi_0$ and $(\hat{s}_{\mathbf{k}}^x)^*\psi_0$ are in the domain of H this definition is correct Short calculation shows that:

$$\mathcal{D}_{\mathbf{k}} = \frac{1}{2} \sum_{n > 0} \left[|(\hat{s}_{\mathbf{k}}^x \psi_0 | \psi_n)|^2 + |((\hat{s}_{\mathbf{k}}^x)^* \psi_0 | \psi_n)|^2 \right] (E_n - E_0) \quad (56)$$

We shall also use the *susceptibility* which is given by:

$$\chi_{\mathbf{k}} = \frac{1}{2} \sum_{n > 0} \left[|(\hat{s}_{\mathbf{k}}^x \psi_0 | \psi_n)|^2 + |((\hat{s}_{\mathbf{k}}^x)^* \psi_0 | \psi_n)|^2 \right] \frac{1}{E_n - E_0} \quad (57)$$

Clearly $\chi_{\mathbf{k}} < \infty$ due to (54) and the fact that $\lim E_n = +\infty$.

For a positive integer $n > 0$ let

$$a_n := \sqrt{|(\hat{s}_{\mathbf{k}}^x \psi_0 | \psi_n)|^2 + |((\hat{s}_{\mathbf{k}}^x)^* \psi_0 | \psi_n)|^2} \frac{1}{\sqrt{E_n - E_0}},$$

$$b_n := \sqrt{\left[|(\hat{s}_{\mathbf{k}}^x \psi_0 | \psi_n)|^2 + |((\hat{s}_{\mathbf{k}}^x)^* \psi_0 | \psi_n)|^2 \right] (E_n - E_0)}.$$

Then $g_{\mathbf{k}}^2 = (\sum a_n b_n)^2$ and (by Schwarz inequality):

$$g_{\mathbf{k}}^2 = \left(\sum a_n b_n \right)^2 \leq \left(\sum a_n^2 \right) \left(\sum b_n^2 \right) = \chi_{\mathbf{k}} \mathcal{D}_{\mathbf{k}}$$

Notice, that since $\hat{s}_{\mathbf{k}}^x$ is a multiplication by a smooth function and H is a second order differential operator, the double commutator appearing in (55) is an operator of multiplication by a smooth function, so it is bounded. In fact by a direct calculation one gets

$$\mathcal{D}_{\mathbf{k}} = \frac{1}{2} \langle [(\hat{s}_{\mathbf{k}}^x, H), (\hat{s}_{\mathbf{k}}^x)^*] \rangle \leq \frac{1}{4I}. \quad (58)$$

Using this estimate we obtain the inequality:

$$g_{\mathbf{k}}^2 \leq \chi_{\mathbf{k}} \cdot \mathcal{D}_{\mathbf{k}} \leq \frac{\chi_{\mathbf{k}}}{4I}. \quad (59)$$

This shows that an upper bound for $\chi_{\mathbf{k}}$ implies the upper bound for $g_{\mathbf{k}}$.

3.4 Reflection Positivity arguments

Let us now proceed along the general line of RP arguments. We perturb the Hamiltonian in the analogous manner as it was done in the case of positive temperatures [5], [10]. To do it, let us first modify the original Hamiltonian (43) by a constant (so irrelevant) term introduced by the potential:

$$V = \frac{J}{2} \sum_{\langle \mathbf{x}\mathbf{y} \rangle} [(\cos \varphi_{\mathbf{x}} - \cos \varphi_{\mathbf{y}})^2 + (\sin \varphi_{\mathbf{x}} - \sin \varphi_{\mathbf{y}})^2] = V_0 + J \sum_{\langle \mathbf{x}\mathbf{y} \rangle} 1 \quad (60)$$

Now, for a function $b : \Lambda \ni \mathbf{x} \mapsto b_{\mathbf{x}} \in \mathbb{C}$ defined on sites, let us consider the perturbed potential $V(b)$:

$$V(b) = \frac{J}{2} \sum_{\langle \mathbf{x}\mathbf{y} \rangle} [|\cos \varphi_{\mathbf{x}} - b_{\mathbf{x}} - \cos \varphi_{\mathbf{y}} + b_{\mathbf{y}}|^2 + (\sin \varphi_{\mathbf{x}} - \sin \varphi_{\mathbf{y}})^2] \quad (61)$$

and the perturbed hamiltonian

$$H(b) := T + \hat{V}(b). \quad (62)$$

Let us remark, that if functions b and b' differ by a constant function then $H(b) = H(b')$, in particular for b being a constant $H(b) = H(0)$.

Clearly the perturbed Hamiltonian satisfies all assumptions of Thm 3.1. We will use the Reflection Positivity and the operator inequality in the form given in (34) to prove the following:

Theorem 3.5 *Let $E_0(b)$ be the ground state energy of the perturbed hamiltonian (62). Then:*

$$E_0(b) \geq E_0(0) =: E_0 \quad (63)$$

The rest of the subsection is devoted to the proof of above theorem. Let us recall the definition of Λ (41):

$$\Lambda := \{\mathbf{x} \in \mathbb{Z}^d : -N + 1 \leq x_i \leq N, i = 1, \dots, d\} \quad (64)$$

We divide the system into two identical subsystems Λ_L, Λ_R so that $\Lambda = \Lambda_L \cup \Lambda_R$; Λ_L is an mirror image of Λ_R under reflection in the Π (hyper) plane given by the equation $x_1 = \frac{1}{2}$. This way, the subset Λ_L contains all sites, where the first coordinate is negative or 0, and Λ_R – sites, where it is positive.

The Hilbert space \mathcal{H}_{Λ} of states of the whole system is a tensor product of two spaces: $\mathcal{H}_{\Lambda} = \mathcal{L} \otimes \mathcal{R}$, where \mathcal{L} (\mathcal{R}) is a space of states of subsystem defined on Λ_L (Λ_R).

For a function $b : \Lambda \rightarrow \mathbb{C}$ define:

$$\begin{cases} b_L = b & \text{on } \Lambda_L, & b_L \text{ is defined on } \Lambda_R & \text{by the mirror symmetry} \\ b_R = b & \text{on } \Lambda_R, & b_R \text{ is defined on } \Lambda_L & \text{by the mirror symmetry} \end{cases} \quad (65)$$

(the standard trick in RP). Our first step to prove inequality (63) is the following :

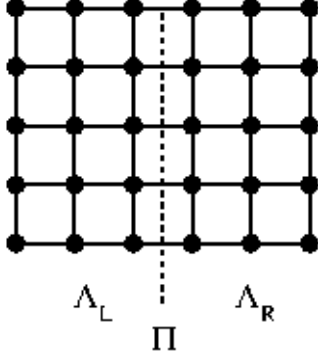


Figure 1: Division of the system into two identical subsystems Λ_L, Λ_R by the symmetry plane Π . The illustration concerns open boundary conditions. Let us stress that in the paper we have to do with periodic boundary conditions, where analogous division can be made.

Lemma 3.6 *Let $b : \Lambda \rightarrow \mathbb{C}$ be a function and b_L, b_R be related to b as in (65). Let hamiltonians $H(b), H(b_R), H(b_L)$ be defined by (62) and $E_0(b), E_0(b_R), E_0(b_L)$ denote their ground state energies respectively. Then:*

$$2E_0(b) \geq E_0(b_R) + E_0(b_L) \quad (66)$$

We move the proof of this lemma to the end of this subsection.

Now, we will apply RP arguments to show how the inequality (66) implies (63). For a given function $b : \Lambda \rightarrow \mathbb{C}$ let us call a *non-zero bond* for b a pair of nearest neighbours (\mathbf{x}, \mathbf{y}) with $b(\mathbf{x}) \neq b(\mathbf{y})$. Clearly, if $b \neq \text{const}$ then there are $l > 0$ non-zero bonds for b and we can choose the symmetry plane, which crosses at least one of them. Notice that if l_L, l_R are number of non-zero bonds for b_L and b_R respectively then, we have:

$$l_L + l_R < 2l;$$

i.e. at least one of the numbers (l_L, l_R) is less than l . Therefore for any non-constant b we can pass to a constant one by applying finitely many replacements $b \rightarrow b_L$ or $b \rightarrow b_R$ (for different symmetry planes). For a given lattice Λ the maximal number of steps is bounded independently of b – let K denotes this bound.

Our modified hamiltonian (62) is a *positive* operator, so the set $\{E_0(b)\}$ is bounded from below and let us define $\mathbf{E} := \inf \{E_0(b)\}$. From the inequality (66) it follows that for $\epsilon > 0$:

$$\text{if } \mathbf{E} \leq E_0(b) \leq \mathbf{E} + \epsilon \text{ then } \mathbf{E} \leq E_0(b_L) \leq \mathbf{E} + 2\epsilon \text{ and } \mathbf{E} \leq E_0(b_R) \leq \mathbf{E} + 2\epsilon \quad (67)$$

So let $\epsilon > 0$ be given. There exists \tilde{b} with $\mathbf{E} \leq E_0(\tilde{b}) \leq \mathbf{E} + 2^{-K}\epsilon$; replacing successively \tilde{b} by \tilde{b}_L or \tilde{b}_R and applying (67) we obtain $\tilde{b}_0 = \text{const}$ with

$$\mathbf{E} \leq E_0(\tilde{b}_0) \leq \mathbf{E} + \epsilon;$$

but since $E_0(\tilde{b}_0) = E_0(0) = E_0$ it follows that $\mathbf{E} = E_0$ – so to prove the inequality (63) it remains to prove the lemma 3.6.

Proof of the lemma 3.6: Consider the perturbed hamiltonian $H(b) = T + \hat{V}(b)$, where $V(b)$ is given by (62). The proof consists of two steps: the first is rather involved – by using the lemma (2.8) and prop (2.10) we will show that for a *smooth* function $\Psi \in \mathcal{L} \otimes \mathcal{R}$, there exist $\Psi_L, \Psi_R \in \mathcal{L} \otimes \mathcal{R}$ such that:

$$2(\Psi | H(b)\Psi) \geq (\Psi_L | H(b_L)\Psi_L) + (\Psi_R | H(b_R)\Psi_R) \quad (68)$$

Since, by the *variational principle* $(\Psi_L | H(b_L)\Psi_L) \geq E_0(b_L)$ and $(\Psi_R | H(b_R)\Psi_R) \geq E_0(b_R)$, we get:

$$(\Psi_L | H(b_L)\Psi_L) \geq E_0(b_L) + E_0(b_R).$$

For $\Psi = \psi_0$ – the ground state of $H(b)$, we obtain the inequality (66).

To simplify notation let us define $a_{\mathbf{x}}(b) := \cos \varphi_{\mathbf{x}} - b_{\mathbf{x}}$; we will write just $a_{\mathbf{x}}$ if it is clear what is b . Now, $V(b)$ reads:

$$V(b) = \frac{J}{2} \sum_{\langle \mathbf{x}\mathbf{y} \rangle} |a_{\mathbf{x}} - a_{\mathbf{y}}|^2 + (\sin \varphi_{\mathbf{x}} - \sin \varphi_{\mathbf{y}})^2$$

Let us define subsets $B_L \subset \Lambda_L$, $B_R \subset \Lambda_R$:

$$\begin{aligned} B_L &:= \{\mathbf{x} \in \Lambda_L : \text{the first coordinate of } \mathbf{x} \text{ is } 0 \text{ or } -N+1\} \\ B_R &:= \{\mathbf{x} \in \Lambda_R : \text{the first coordinate of } \mathbf{x} \text{ is } 1 \text{ or } N\} \end{aligned} \quad (69)$$

Notice that B_L (B_R) is the subset of those elements in Λ_L (Λ_R) which have (some of) their nearest neighbours in Λ_R (Λ_L). The potential $V(b)$ can be written as:

$$\begin{aligned} V(b) &= \frac{J}{2} \sum_{\langle \mathbf{x}\mathbf{y} \rangle \subset L} |a_{\mathbf{x}} - a_{\mathbf{y}}|^2 + (\sin \varphi_{\mathbf{x}} - \sin \varphi_{\mathbf{y}})^2 + \frac{J}{2} \sum_{\langle \mathbf{x}\mathbf{y} \rangle \subset R} |a_{\mathbf{x}} - a_{\mathbf{y}}|^2 + (\sin \varphi_{\mathbf{x}} - \sin \varphi_{\mathbf{y}})^2 + \\ &+ \frac{J}{2} \sum_{\mathbf{x} \in B_L} |a_{\mathbf{x}} - a_{\mathbf{x}'}|^2 + (\sin \varphi_{\mathbf{x}} - \sin \varphi_{\mathbf{x}'})^2, \end{aligned}$$

where \mathbf{x}' is the image of \mathbf{x} by the reflection across the hyperplane Π . The last term of the sum above we write as:

$$\begin{aligned} \frac{J}{2} \sum_{\mathbf{x} \in B_L} |a_{\mathbf{x}} - a_{\mathbf{x}'}|^2 + (\sin \varphi_{\mathbf{x}} - \sin \varphi_{\mathbf{x}'})^2 &= \frac{J}{2} \sum_{\mathbf{x} \in B_L} (|a_{\mathbf{x}}|^2 + \sin^2 \varphi_{\mathbf{x}}) + \frac{J}{2} \sum_{\mathbf{x} \in B_L} (|a_{\mathbf{x}'}|^2 + \sin^2 \varphi_{\mathbf{x}'}) + \\ &- \frac{J}{2} \sum_{\mathbf{x} \in B_L} (\overline{a_{\mathbf{x}}} a_{\mathbf{x}'} + a_{\mathbf{x}} \overline{a_{\mathbf{x}'}} + 2 \sin \varphi_{\mathbf{x}} \sin \varphi_{\mathbf{x}'}) \end{aligned}$$

Let us define functions:

$$V_L(b) := \frac{J}{2} \sum_{\langle \mathbf{x}\mathbf{y} \rangle \subset L} |a_{\mathbf{x}}(b) - a_{\mathbf{y}}(b)|^2 + (\sin \varphi_{\mathbf{x}} - \sin \varphi_{\mathbf{y}})^2 + \frac{J}{2} \sum_{\mathbf{x} \in B_L} |a_{\mathbf{x}}(b)|^2 + \sin^2 \varphi_{\mathbf{x}} \quad (70)$$

$$V_R(b) := \frac{J}{2} \sum_{\langle \mathbf{x}\mathbf{y} \rangle \subset R} |a_{\mathbf{x}}(b) - a_{\mathbf{y}}(b)|^2 + (\sin \varphi_{\mathbf{x}} - \sin \varphi_{\mathbf{y}})^2 + \frac{J}{2} \sum_{\mathbf{x} \in B_L} |a_{\mathbf{x}'}(b)|^2 + \sin^2 \varphi_{\mathbf{x}'} \quad (71)$$

$$V_I(b) := -\frac{J}{2} \sum_{\mathbf{x} \in B_L} (\overline{a_{\mathbf{x}}} a_{\mathbf{x}'} + a_{\mathbf{x}} \overline{a_{\mathbf{x}'}} + 2 \sin \varphi_{\mathbf{x}} \sin \varphi_{\mathbf{x}'}) \quad (72)$$

Using above notation we can write the potential $V(b)$ as: $V(b) = V_L(b) + V_R(b) + V_I(b)$ and the corresponding operator $\hat{V}(b)$ as $\hat{V}(b) = \widehat{V_L}(b) \otimes I + I \otimes \widehat{V_R}(b) + \widehat{V_I}(b)$, where

$$\widehat{V_I}(b) := -\frac{J}{2} \sum_{\mathbf{x} \in B_L} (\overline{a_{\mathbf{x}}} \otimes a_{\mathbf{x}'} + a_{\mathbf{x}} \otimes \overline{a_{\mathbf{x}'}} + 2 \sin \varphi_{\mathbf{x}} \otimes \sin \varphi_{\mathbf{x}'}). \quad (73)$$

For the kinetic term T we have $T = T_L \otimes I + I \otimes T_R$.

Now, Let us choose an orthonormal basis $\Gamma := \{\psi_{\gamma}\}$ in \mathcal{L} consisting of eigenvectors of T_L : $T_L \psi_{\gamma} = t_{\gamma} \psi_{\gamma}$ and the corresponding involution J_{Γ} . Note that since T_L is self-adjoint it commutes with J_{Γ} :

$$J_{\Gamma} T_L J_{\Gamma} = T_L. \quad (74)$$

Let $U : \mathcal{L} \rightarrow \mathcal{R}$ be a unitary operator and $\phi_{\gamma} := U \psi_{\gamma}$ be the corresponding basis in \mathcal{R} .

Let us also assume that

$$T_R = U T_L U^* \quad (75)$$

That means, in particular, that $\phi_{\gamma} \in D(T_R)$ and $T_R \phi_{\gamma} = t_{\gamma} \phi_{\gamma}$

Let $\Psi =: \hat{\Gamma}(c) \in \mathcal{L} \otimes \mathcal{R}$ be a smooth function; then it belongs to domains of T^k , $(T_L \otimes I)^k$, $(I \otimes T_R)^k$ for $k = 1, 2, 3, \dots$ and, using (70), (71) and (72), we can write:

$$\begin{aligned} 2(\Psi | H(b) \Psi) &= 2(\Psi | T \Psi) + 2(\Psi | \widehat{V}(b) \Psi) = \\ &= (\Psi | (T_L \otimes I) \Psi) + (\Psi | (T_L \otimes I) \Psi) + \end{aligned} \quad (76)$$

$$+ (\Psi | (I \otimes T_R) \Psi) + (\Psi | (I \otimes T_R) \Psi) + \quad (77)$$

$$+ (\Psi | (\widehat{V_L}(b) \otimes I) \Psi) + (\Psi | (\widehat{V_L}(b) \otimes I) \Psi) + \quad (78)$$

$$+ (\Psi | (I \otimes \widehat{V_R}(b)) \Psi) + (\Psi | (I \otimes \widehat{V_R}(b)) \Psi) + \quad (79)$$

$$+ 2(\Psi | \widehat{V_I}(b) \Psi) \quad (80)$$

Now, due to the lemma 2.8 and proposition 2.10, we are going to rewrite various terms appearing in this equality in different form.

By formulae (35) and (75) for terms in (76) we have:

$$\begin{aligned}(\Psi | (\mathbf{T}_L \otimes I) \Psi) &= \left(\tilde{\Gamma}(U|c|) | (\mathbf{T}_L \otimes I) \tilde{\Gamma}(U|c|) \right) \\(\Psi | (\mathbf{T}_L \otimes I) \Psi) &= \left(\tilde{\Gamma}(U|c|) | (I \otimes \mathbf{T}_R) \tilde{\Gamma}(U|c|) \right); \end{aligned}$$

Using (36) with $d = J_\Gamma c^* J_\Omega$ (then $|d| = J_\Omega |c^*| J_\Omega$ and $\tilde{\Omega}(U^*|d|) = \tilde{\Gamma}(|c^*|U)$) and (75) for terms in (77):

$$\begin{aligned}(\Psi | (I \otimes \mathbf{T}_R) \Psi) &= \left(\tilde{\Gamma}(|c^*|U) | (I \otimes \mathbf{T}_R) \tilde{\Gamma}(|c^*|U) \right) \\(\Psi | (I \otimes \mathbf{T}_R) \Psi) &= \left(\tilde{\Gamma}(|c^*|U) | (\mathbf{T}_L \otimes I) \tilde{\Gamma}(|c^*|U) \right) \end{aligned}$$

For terms appearing in (78) and (79) by the use of (29), (30), (28) and (31) we get:

$$\begin{aligned}(\Psi | (\widehat{V}_L(b) \otimes I) \Psi) &= \left(\tilde{\Gamma}(U|c|) | (\widehat{V}_L(b) \otimes I) \tilde{\Gamma}(U|c|) \right) \\(\Psi | (\widehat{V}_L(b) \otimes I) \Psi) &= \left(\tilde{\Gamma}(U|c|) | (I \otimes U J_\Gamma (\widehat{V}_L(b))^* J_\Gamma U^*) \tilde{\Gamma}(U|c|) \right) \\(\Psi | (I \otimes \widehat{V}_R(b)) \Psi) &= \left(\tilde{\Gamma}(|c^*|U) | (I \otimes \widehat{V}_R(b)) \tilde{\Gamma}(|c^*|U) \right) \\(\Psi | (I \otimes \widehat{V}_R(b)) \Psi) &= \left(\tilde{\Gamma}(|c^*|U) | (U^* J_\Omega (\widehat{V}_R(b))^* J_\Omega U \otimes I) \tilde{\Gamma}(|c^*|U) \right) \end{aligned}$$

Now we are going to use the inequality (34) to estimate the term in (80).

Since $(\Psi | (\overline{a_{\mathbf{x}}} \otimes a_{\mathbf{x}'} + a_{\mathbf{x}} \otimes \overline{a_{\mathbf{x}'}})) \Psi$ is *real* we have:

$$\begin{aligned}(\Psi | (\overline{a_{\mathbf{x}}} \otimes a_{\mathbf{x}'} + a_{\mathbf{x}} \otimes \overline{a_{\mathbf{x}'}})) \Psi &\leq |(\Psi | (\overline{a_{\mathbf{x}}} \otimes a_{\mathbf{x}'} + a_{\mathbf{x}} \otimes \overline{a_{\mathbf{x}'}})) \Psi| \leq \\ &\leq |(\Psi | (\overline{a_{\mathbf{x}}} \otimes a_{\mathbf{x}'}) \Psi)| + |(\Psi | (a_{\mathbf{x}} \otimes \overline{a_{\mathbf{x}'}})) \Psi)| \end{aligned}$$

Applying the inequality (34) to each of two terms we get:

$$\begin{aligned}2 |(\Psi | (\overline{a_{\mathbf{x}}} \otimes a_{\mathbf{x}'}) \Psi)| &\leq \left(\tilde{\Gamma}(U|c|) | (\overline{a_{\mathbf{x}}} \otimes U J_\Gamma \overline{a_{\mathbf{x}}} J_\Gamma U^*) \tilde{\Gamma}(U|c|) \right) + \\ &+ \left(\tilde{\Gamma}(|c^*|U) | (U^* J_\Omega a_{\mathbf{x}'} J_\Omega U \otimes a_{\mathbf{x}'}) \tilde{\Gamma}(|c^*|U) \right) \end{aligned}$$

and

$$\begin{aligned}2 |(\Psi | (a_{\mathbf{x}} \otimes \overline{a_{\mathbf{x}'}})) \Psi)| &\leq \left(\tilde{\Gamma}(U|c|) | (a_{\mathbf{x}} \otimes U J_\Gamma a_{\mathbf{x}} J_\Gamma U^*) \tilde{\Gamma}(U|c|) \right) + \\ &+ \left(\tilde{\Gamma}(|c^*|U) | (U^* J_\Omega \overline{a_{\mathbf{x}'}} J_\Omega U \otimes \overline{a_{\mathbf{x}'}})) \tilde{\Gamma}(|c^*|U) \right) \end{aligned}$$

Similarly, since $2 (\Psi | (\sin \varphi_{\mathbf{x}} \otimes \sin \varphi_{\mathbf{x}'})) \Psi$ is *real*:

$$\begin{aligned}2 (\Psi | (\sin \varphi_{\mathbf{x}} \otimes \sin \varphi_{\mathbf{x}'})) \Psi &\leq 2 |(\Psi | (\sin \varphi_{\mathbf{x}} \otimes \sin \varphi_{\mathbf{x}'})) \Psi| \leq \\ &\leq \left(\tilde{\Gamma}(U|c|) | (\sin \varphi_{\mathbf{x}} \otimes U J_\Gamma \sin \varphi_{\mathbf{x}} J_\Gamma U^*) \tilde{\Gamma}(U|c|) \right) + \\ &+ \left(\tilde{\Gamma}(|c^*|U) | (U^* J_\Omega \sin \varphi_{\mathbf{x}'} J_\Omega U \otimes \sin \varphi_{\mathbf{x}'}) \tilde{\Gamma}(|c^*|U) \right) \end{aligned}$$

Adding these three inequalities and multiplying by a *negative number* $(-J/2)$ we obtain the estimate for (80):

$$2 (\Psi | \widehat{V}_I(b) \Psi) \geq \left(\tilde{\Gamma}(U|c|) | K_L \tilde{\Gamma}(U|c|) \right) + \left(\tilde{\Gamma}(|c^*|U) | K_R \tilde{\Gamma}(|c^*|U) \right),$$

where operators K_L and K_R are defined by:

$$K_L := -\frac{J}{2} \sum_{\mathbf{x} \in B_L} [\overline{a_{\mathbf{x}}} \otimes U J_\Gamma \overline{a_{\mathbf{x}}} J_\Gamma U^* + a_{\mathbf{x}} \otimes U J_\Gamma a_{\mathbf{x}} J_\Gamma U^* + 2 \sin \varphi_{\mathbf{x}} \otimes U J_\Gamma \sin \varphi_{\mathbf{x}} J_\Gamma U^*] \quad (81)$$

$$K_R := -\frac{J}{2} \sum_{\mathbf{x} \in B_L} [U^* J_\Omega a_{\mathbf{x}'} J_\Omega U \otimes a_{\mathbf{x}'} + U^* J_\Omega \overline{a_{\mathbf{x}'}} J_\Omega U \otimes \overline{a_{\mathbf{x}'}} + 2 U^* J_\Omega \sin \varphi_{\mathbf{x}'} J_\Omega U \otimes \sin \varphi_{\mathbf{x}'}] \quad (82)$$

Putting all together we obtain the following inequality :

$$2(\Psi | H(b)\Psi) \geq (\Psi_L | H_1 \Psi_L) + (\Psi_R | H_2 \Psi_R), \text{ where} \quad (83)$$

$$\Psi_L := \tilde{\Gamma}(U|c), \quad \Psi_R := \tilde{\Gamma}(|c^*|U) \quad (84)$$

$$H_1 := T_L \otimes I + I \otimes T_R + \widehat{V}_L(b) \otimes I + I \otimes U J_\Gamma(\widehat{V}_L(b))^* J_\Gamma U^* + K_L \quad (85)$$

$$H_2 := T_L \otimes I + I \otimes T_R + U^* J_\Omega(\widehat{V}_R(b))^* J_\Omega U \otimes I + I \otimes \widehat{V}_R(b) + K_R \quad (86)$$

We would like to have $H_1 = H(b_L)$, $H_2 = H(b_R)$. Notice that the H_1 and H_2 depend only on composition $U J_\Gamma$ (by (27) $J_\Omega U = U J_\Gamma$); remember also that we have assumed in (75) that $T_R = U T_L U^*$.

Let $M : \Lambda \rightarrow \Lambda$ be the reflection across our (hyper)plane Π and let $U_M : \mathcal{L} \rightarrow \mathcal{R}$ be the unitary defined by

$$(U_M \psi)(\varphi_{\mathbf{y}_1}, \varphi_{\mathbf{y}_2}, \dots) := \psi(\varphi_{\mathbf{x}_1}, \varphi_{\mathbf{x}_2}, \dots) \quad (87)$$

$$\mathbf{y}_i := M(\mathbf{x}_i), \quad \mathbf{x}_i \in \Lambda_L, \quad i = 1, \dots, |\Lambda_L|,$$

and J_0 be the complex conjugation (we will use J_0 for conjugations on \mathcal{L} and \mathcal{R}). It is clear that

$$J_0 U_M = U_M J_0, \quad T_L J_0 = J_0 T_L, \quad T_R J_0 = J_0 T_R,$$

Define the unitary $U := J_0 U_M J_\Gamma = U_M J_0 J_\Gamma$; then

$$U T_L U^* = U_M J_0 J_\Gamma T_L J_\Gamma J_0 U_M^* = U_M T_L U_M^* = T_R \text{ and } U J_\Gamma = J_0 U_M = U_M J_0. \quad (88)$$

We will analyze H_1 . Clearly we have $V_L(b) = V_L(b_L)$; since the function $V_L(b)$ is real $(\widehat{V}_L(b))^* = \widehat{V}_L(b)$ and $J_0 \widehat{V}_L(b) J_0 = \widehat{V}_L(b)$. Therefore

$$U J_\Gamma(\widehat{V}_L(b))^* J_\Gamma U^* = U_M \widehat{V}_L(b) U_M^* = \widehat{V}_R(b_L)$$

This way we obtain:

$$H_1 = T + \widehat{V}_L(b_L) \otimes I + I \otimes \widehat{V}_R(b_L) + K_L$$

Let us show that K_L given by (81) is equal $\widehat{V}_I(b_L)$ defined in (73):

$$\begin{aligned} U J_\Gamma \overline{a_{\mathbf{x}}(b)} J_\Gamma U^* &= U_M J_0 \overline{(\cos \varphi_{\mathbf{x}} - b_{\mathbf{x}})} J_0 U_M^* = U_M (\cos \varphi_{\mathbf{x}} - b_{\mathbf{x}}) U_M = \cos \varphi_{\mathbf{x}'} - b_L(\mathbf{x}') = \\ &= a_{\mathbf{x}'}(b_L), \end{aligned}$$

where $\mathbf{x}' := M(\mathbf{x})$ (because for $\mathbf{x} \in \Lambda_L$ $b(\mathbf{x}) = b_L(\mathbf{x}) = b_L(\mathbf{x}')$); the next term:

$$U J_\Gamma a_{\mathbf{x}}(b) J_\Gamma U^* = U_M \overline{a_{\mathbf{x}}(b)} U_M^* = \overline{a_{\mathbf{x}'}(b_L)}$$

and finally

$$U J_\Gamma \sin \varphi_{\mathbf{x}} J_\Gamma U^* = U_M \sin \varphi_{\mathbf{x}} U_M^* = \sin \varphi_{\mathbf{x}'}$$

So we get:

$$K_L = -\frac{J}{2} \sum_{\mathbf{x} \in B_L} \left[\overline{a_{\mathbf{x}}(b_L)} \otimes a_{\mathbf{x}'}(b_L) + a_{\mathbf{x}}(b_L) \otimes \overline{a_{\mathbf{x}'}(b_L)} + 2 \sin \varphi_{\mathbf{x}} \otimes \sin \varphi_{\mathbf{x}'} \right]$$

and this is $\widehat{V}_I(b_L)$, so really $H_1 = H(b_L)$; in the same way one gets $H_2 = H(b_R)$.

This way the proof of the inequality (68) and the lemma 3.6 is completed as well as the proof of the inequality (63). \blacksquare

3.5 Estimations giving LRO

In this subsection we finally complete the proof of LRO, i.e. we show the inequality (52). The inequality (63) implies that

$$\left. \frac{d^2 E(\lambda b)}{d\lambda^2} \right|_{\lambda=0} \geq 0 \quad (89)$$

for an *arbitrary* b . It turns out that if we take b being *plane wave* with the wave vector \mathbf{k} then we get (52). Let us present calculations in more details.

For the moment, let us keep the b function being arbitrary. Write the perturbed Hamiltonian (62) with the b function rescaled by a factor $\lambda \in \mathbb{R}$:

$$H(\lambda b) = H(0) + \lambda H'(b) + \lambda^2 C(b), \quad (90)$$

where

$$H'(b) := -JRe \left(\sum_{\langle \mathbf{x}\mathbf{y} \rangle} (\cos \varphi_{\mathbf{x}} - \cos \varphi_{\mathbf{y}})(b_{\mathbf{x}} - b_{\mathbf{y}}) \right), \quad C(b) := \frac{J}{2} \sum_{\langle \mathbf{x}\mathbf{y} \rangle} |b_{\mathbf{x}} - b_{\mathbf{y}}|^2 \quad (91)$$

Let $E_0(\lambda b)$ be the ground state energy of the operator $H(\lambda b)$, and $\overset{(2)}{\Delta} E_0(b)$ – the correction to ground state energy in the second order perturbation theory for the Hamiltonian $H(0) + \lambda H'(b)$, i.e.

$$\overset{(2)}{\Delta} E_0(b) = \sum_{n>0} \frac{|\langle \psi_n | H'(b) \psi_0 \rangle|^2}{E_0 - E_n}$$

Therefore:

$$\left. \frac{d^2 E(\lambda b)}{d\lambda^2} \right|_{\lambda=0} = \overset{(2)}{\Delta} E_0(b) + 2C(b) \geq 0. \quad (92)$$

Choose now the b function as

$$b_{\mathbf{x}} = \frac{1}{\sqrt{|\Lambda|}} e^{i\mathbf{k} \cdot \mathbf{x}} \quad (93)$$

With such a choice (91) reads:

$$C(b) := \frac{J}{2|\Lambda|} \sum_{\langle \mathbf{x}\mathbf{y} \rangle} |e^{i\mathbf{k} \cdot \mathbf{x}} - e^{i\mathbf{k} \cdot \mathbf{y}}|^2 = J\mathcal{E}(\mathbf{k}) \quad (94)$$

and, since

$$\frac{1}{\sqrt{|\Lambda|}} \sum_{\langle \mathbf{x}\mathbf{y} \rangle} (\cos \varphi_{\mathbf{x}} - \cos \varphi_{\mathbf{y}})(e^{i\mathbf{k} \cdot \mathbf{x}} - e^{i\mathbf{k} \cdot \mathbf{y}}) = s_{\mathbf{k}}^x \sum_{j=1}^d (2 - 2 \cos \mathbf{k}_j) \equiv 2s_{\mathbf{k}}^x \mathcal{E}(\mathbf{k}), \quad (95)$$

$$H'(b) = -2J\mathcal{E}(\mathbf{k})Re(s_{\mathbf{k}}^x),$$

Therefore by (92):

$$-4J^2\mathcal{E}^2(\mathbf{k}) \sum_{n>0} \frac{|\langle \psi_n | Re(s_{\mathbf{k}}^x) \psi_0 \rangle|^2}{E_n - E_0} + 2J\mathcal{E}(\mathbf{k}) \geq 0$$

By the similar computations, replacing b by ib we obtain: $H'(ib) = 2J\mathcal{E}(\mathbf{k})Im(s_{\mathbf{k}}^x)$ and

$$-4J^2\mathcal{E}^2(\mathbf{k}) \sum_{n>0} \frac{|\langle \psi_n | Im(s_{\mathbf{k}}^x) \psi_0 \rangle|^2}{E_n - E_0} + 2J\mathcal{E}(\mathbf{k}) \geq 0$$

Adding these inequalities we get (remember $J\mathcal{E}(\mathbf{k}) \geq 0$):

$$\sum_{n>0} \frac{|\langle \psi_n | Re(s_{\mathbf{k}}^x) \psi_0 \rangle|^2}{E_n - E_0} + \frac{|\langle \psi_n | Im(s_{\mathbf{k}}^x) \psi_0 \rangle|^2}{E_n - E_0} \leq \frac{1}{J\mathcal{E}(\mathbf{k})}.$$

For complex numbers $\alpha := \langle \psi_n | s_{\mathbf{k}}^x \psi_0 \rangle$ and $\beta := \langle \psi_n | (s_{\mathbf{k}}^x)^* \psi_0 \rangle$, by the parallelogram law : $|\alpha + \beta|^2 + |\alpha - \beta|^2 = 2(|\alpha|^2 + |\beta|^2)$, we get

$$|\langle \psi_n | Re(s_{\mathbf{k}}^x) \psi_0 \rangle|^2 + |\langle \psi_n | Im(s_{\mathbf{k}}^x) \psi_0 \rangle|^2 = \frac{1}{2} (|\langle \psi_n | s_{\mathbf{k}}^x \psi_0 \rangle|^2 + |\langle \psi_n | (s_{\mathbf{k}}^x)^* \psi_0 \rangle|^2),$$

and finally:

$$\frac{1}{2} \sum_{n>0} \frac{|\langle \psi_n | s_{\mathbf{k}}^x \psi_0 \rangle|^2}{E_n - E_0} + \frac{|\langle \psi_n | (s_{\mathbf{k}}^x)^* \psi_0 \rangle|^2}{E_n - E_0} \leq \frac{1}{J\mathcal{E}(\mathbf{k})}.$$

The LHS of this inequality is just susceptibility $\chi_{\mathbf{k}}$, compare (57), so we have $\chi_{\mathbf{k}} \leq \frac{1}{J\mathcal{E}(\mathbf{k})}$. Using (58) and (59), we obtain the estimate (52):

$$g_{\mathbf{k}}^2 \leq \chi_{\mathbf{k}} \cdot \mathcal{D}_{\mathbf{k}} \leq \chi_{\mathbf{k}} \frac{1}{4I} \leq \frac{1}{4IJ\mathcal{E}(\mathbf{k})}$$

The proof of theorem 3.3 is complete. ■

4 Summary

We have extended the Kennedy-Lieb-Shastry-Schupp matrix inequality to the case where matrices are replaced by certain infinite dimensional operators. Similar result has been proven in [5] for another matrix inequality – the DLS lemma, which is crucial for the proof of occurrence of LRO in the system of interacting rotors in low temperatures in $d \geq 3$.

With the use of this inequality and Reflection Positivity technology, we have formulated sufficient condition (44) for ordering in the ground state of the system of interacting rotors. In particular, the LRO is present in $d \geq 2$ for sufficiently large value of IJ . This way, we have shown the occurrence of the LRO in the ground state of interacting rotor systems in a direct manner. In the paper [10], analogous result has been proven by Reflection Positivity technique, but without checking some assumptions (validity of certain limiting procedure). Our present approach does not suffer from this drawback. This result has also been obtained in non-rigorous way by scaling and field-theoretic arguments [12]

We are convinced that our result can be extended to other rotor systems: other (bipartite) lattices and larger space of internal degrees of freedom, for instance, for $O(n)$ systems.

One can pose the problem concerning the occurrence the ordering in opposite situation, i.e. for quantity IJ being small. To our best knowledge, this is an open question. One can suspect that the LRO should be absent. Such expectation is motivated by the paper [21], where somewhat similar result has been proved: There is no ordering in the anharmonic crystal model provided mass of the oscillator is sufficiently small.

There are numerous interesting rotor-like systems, which do *not* fulfill conditions allowing an application of Reflection Positivity techniques. One of most important of them, is the lattice system of interacting bosons (for instance, the Bose Hubbard model). The Hamiltonian of this system, written in the language of coherent states, becomes the Hamiltonian of interacting rotors of the form (43), plus one term more (see, for instance [22]). This last term spoils the Reflection Positivity, and it seems to be not possible to apply these techniques to the analysis of interacting boson systems. (Only exception is the paper [20], where the Bose-Einstein condensation has been proved for hard-core bosons on optical lattice. Here, the term spoiling RP is absent due to the hard-core condition). Here we tackle with the long-standing and important problem: How to extend the range of applicability of Reflection Positivity technique, which works for certain problems, and does not work for apparently very similar ones.

Another interesting problem is the occurrence of the Kosterlitz-Thouless transition [24] in the two-dimensional rotor system. On physical grounds, one can expect occurrence of this transition, at least for large momentum of inertia (the quantum-mechanical rotors should not differ too much from the 2d XY model, for which such a transition has been rigorously proven [25]) However, we are not aware on rigorous results for interacting rotor systems.

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